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Analysis and classification of nonlinear dispersive evolution equations in the potential representation

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Abstract

A potential representation for the subset of travelling solutions of nonlinear dispersive evolution equations is introduced. The procedure involves reduction of a third-order partial differential equation to a first-order ordinary differential equation. The potential representation allows us to deduce certain properties of the solutions without the actual need to solve the underlying evolution equation. In particular, the paper deals with the so-called $K(n, m)$ equations. Starting from their respective potential representations it is shown that these equations can be classified according to a simple point transformation. As a result, e.g., all equations with linear dispersion join the same equivalence class with the Korteweg–deVries equation being its representative, and all soliton solutions of higher order nonlinear equations are thus equivalent to the KdV soliton. Certain equations with both linear and quadratic dispersions can also be treated within this equivalence class.

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1. Introduction

The description of a physical system under extreme conditions, e.g., large amplitude excitations, requires more than a linear theory. However, since dissipation is normally only effective over large timescales, it can be neglected, at least in the first approximation, leaving a nonlinear dispersive partial differential equation for describing the behaviour of such a system. In particular, one typically has to deal with third-order as well as first-order spatial derivatives of a wavefunction u or powers of u , i.e. $(u^m)_{xxx}$, which describe (nonlinear) dispersion, and $(u^n)_x$, which describe nonlinear convection, plus a dynamical term, namely, the time derivative of the wavefunction, u_t . The subscripts denote partial differentiation with respect to the index. The dissipation would correspond to the second-order spatial derivative of the wavefunction.

A special class of nonlinear dispersive evolution equations in $(1 + 1)$ dimensions are the so-called $K(n, m)$ equations, $(u^m)_{xxx} - A(u^n)_x + u_t = 0$ [1]. The most prominent examples of these are the Korteweg–deVries (KdV) equation, $K(2, 1)$, and the modified KdV equation, $K(3, 1)$. These forms are of particular interest because they can be used to describe the motion of stable and localized solitary waves (solitons) that are observed in a variety of physical systems. These forms are also of particular importance for various specialized applications, such as data transfer in optical fibres [2] or as an analytical tool for an explanation of cluster radioactivity [3] or nuclear fission [4].

The considerations in this paper will be carried out for the specific case of travelling wave solutions. It will be shown that this restriction leads to a special representation of nonlinear dispersive evolution equations called the potential representation (this concept has been used previously, for example, in [5, 6]). This representation resembles an energy conservation law with a nonrelativistic kinetic energy term as well as a potential energy. The potential representation, being a first-order ordinary differential equation, constitutes an enormous simplification of the original problem. The gross properties of the solutions can be read directly from the potential function without the actual need to solve a differential equation. The conditions for solitary waves and solitons can thus be easily stated qualitatively. The investigation of the potential picture reveals that for solitons the degree of dispersion may at most be linear, whereas solitary waves with compact support only exist in systems with quadratic dispersion. These results are discussed in more detail for the specific cases of systems that are modelled by $K(n, m)$ equations.

By restricting our consideration to a particular subset of solutions, a $K(n, m)$ equation can be transformed into another $K(N, M)$ equation by means of a simple point transformation. This point transformation defines an equivalence relation between the various $K(n, m)$ equations and in doing so divides these equations into equivalence classes of connected equations.

2. Potential representation

In this paper, the potential representation of a nonlinear evolution equation for travelling waves $u(x, t) = u(x - vt) = u(\xi)$ is defined as follows:

$$(u_\xi)^2 = -\mathcal{F}(u). \quad (1)$$

This notation is inspired by the fact that if ξ and u were identified with *time* and *space*, respectively, the lhs of equation (1) may be associated with a nonrelativistic ‘kinetic’ energy, and, accordingly, the rhs with the negative value of a ‘potential’ energy $\mathcal{F}(u)$. u_ξ is then the ‘velocity’ of a particle moving along the u -axis. The evolution of u proceeds on the zero-energy hypersurface in the phase space belonging to $\mathcal{F}(u)$.

2.1. Reformulation of the nonlinear evolution equation

We consider a general nonlinear dispersive evolution equation of an autonomous one-dimensional non-dissipative dynamical system

$$\mathcal{D}(u)_{xxx} = \mathcal{C}(u)_x - u_t. \quad (2)$$

The ‘dispersion function’ $\mathcal{D}(u)$ and the ‘convection function’ $\mathcal{C}(u)$ have to be at least of class C_3 or C_1 , respectively. We focus on travelling solutions with velocity v so that the partial differential equation (2) is reduced to the ordinary differential equation

$$\mathcal{D}(u)_{\xi\xi\xi} = \mathcal{C}(u)_\xi + vu_\xi. \quad (3)$$

One may combine the terms on the right-hand side writing (the prime indicates differentiation with respect to the argument)

$$\mathcal{C}(u)_\xi + vu_\xi = \mathcal{C}'(u)u_\xi + vu_\xi = \mathcal{V}(u)u_\xi$$

and integrate equation (3) once

$$\mathcal{D}(u)_{\xi\xi} = \int_0^u dt\mathcal{V}(t) - C_1 \tag{4}$$

with an arbitrary constant C_1 . We then use (1) and

$$\mathcal{D}(u)_{\xi\xi} = \mathcal{D}''(u)(u_\xi)^2 + \mathcal{D}'u_{\xi\xi}$$

to find the differential equation defining the potential function $\mathcal{F}(u)$,

$$\frac{1}{2}\mathcal{D}'(u)\mathcal{F}'(u) + \mathcal{D}''(u)\mathcal{F}(u) + \int_0^u dt\mathcal{V}(t) - C_1 = 0. \tag{5}$$

Multiplication with $\mathcal{D}'(u)$ turns (5) into

$$\left(\frac{1}{2}(\mathcal{D}'(u))^2\mathcal{F}(u)\right)' + \mathcal{D}'(u)\int_0^u dt\mathcal{V}(t) - \mathcal{D}'(u)C_1 = 0$$

which yields

$$\mathcal{F}(u) = \frac{2C_1\mathcal{D}(u)}{(\mathcal{D}'(u))^2} + \frac{2C_2}{(\mathcal{D}'(u))^2} - \frac{2}{(\mathcal{D}'(u))^2}\int_0^u dy\mathcal{D}'(y)\int_0^y dt\mathcal{V}(t)$$

C_2 also being an arbitrary constant. Thus casting equation (2) in the potential representation (1) reduces the original problem of integration to a quadrature that can be solved by separation of variables yielding ξ as a function of u :

$$\xi(u) = \int_0^u \frac{dy}{\sqrt{-\mathcal{F}(y)}}. \tag{6}$$

The inversion of this function is a challenging task, which in many cases may not be possible.

In the following, we simplify the considerations to dispersion functions being a simple power of the wavefunction $\mathcal{D}(u) = u^m$. $\mathcal{F}(u)$ is then found to be

$$\mathcal{F}(u) = C_1u^{2-m} + C_2u^{2-2m} - \frac{2}{m^2}u^{3-m}\int_0^1 dy(1-y^m)\mathcal{V}(uy). \tag{7}$$

Finally, we briefly mention for later purposes that the potential function for evolution equations of the form

$$((u + \gamma)^m)_{xxx} = \mathcal{V}(u)u_x$$

with an arbitrary constant γ , is given by

$$\mathcal{F}(u) = C_1(u + \gamma)^{2-m} + C_2(u + \gamma)^{2-2m} - \frac{2}{m^2}(u + \gamma)^{3-m}\int_0^1 dy(1-y^m)\mathcal{V}(uy - (1-y)\gamma). \tag{8}$$

2.2. Analysis of the potential representation

Casting the original nonlinear dispersive evolution equation in the potential representation associates with the wavefunction $u(\xi)$ a spacetime (i.e. $u - \xi$) trajectory of a particle moving in the potential $\mathcal{F}(u)$ with zero total energy. Different types of solutions of the original nonlinear evolution equation can be attributed to different kinds of trajectories in this phase space. For example, closed trajectories in bounded regions of the phase space correspond

to periodic solutions, whereas solitary wave solutions are represented in phase space by separatrix trajectories [7]. In the following, the properties and conditions of solitary waves and of solitary waves with compact support are discussed in more detail. However, the so-called *kinks*, solitary waves which are represented in phase space by separatrix trajectories with two cusps, are not considered.

2.2.1. *Solitons.* In the potential representation, necessary and sufficient conditions for solitary waves read

$$\begin{aligned} \mathcal{F}(a) = \mathcal{F}'(a) = 0 & \quad \mathcal{F}''(a) \leq 0 \\ \mathcal{F}(b) = 0 & \quad \mathcal{F}'(b) \neq 0 \quad a < b. \end{aligned} \quad (9)$$

That is, the potential function must have at least a two-fold zero at a point $u = a$ with negative or vanishing curvature and must have a zero at $b > a$ with nonvanishing slope. In addition, $\mathcal{F}(u)$ must not have singularities in the interval $[a, b]$, and therefore $\mathcal{F}'(b) > 0$. The reasons underlying (9) are as follows: a particle at $u = a$ is at rest (the potential energy is equal to the total energy) and does not experience any force (the gradient of the potential is zero). It takes the particle infinitely long to leave this point. It moves in a positive u -direction, is reflected at $u = b$, moves back in a negative u -direction and reaches a again after another infinite time span, i.e. $\lim_{|\xi| \rightarrow \infty} u(\xi) \rightarrow a$, $u(0) = b$. The simple zero of the potential function, b , thus corresponds to the amplitude of the solitary wave.

Localized solitary waves (solitons) require $a = 0$. For dark solitary waves, i.e. solitary waves with negative amplitude, $\mathcal{F}(u)$ has to fulfil (9) with $b < a$ (the expression *dark soliton* has been introduced in the context of nonlinear optical pulse propagation [8]).

For a more detailed analysis of the implications of (9) let us assume in addition to $\mathcal{D}(u) = u^m$ that $\mathcal{V}(u)$ is continuous in the interval $[a, b]$. Using Weierstrass' theorem one can thus make the general ansatz $\mathcal{V}(u) = \sum a_i u^i$. Here we also assume $a_0 \neq 0$ due to the dynamical term.

One finds that for any $m > 0$, $C_1 = C_2 = 0$, whereas for $m < 0$, C_1 and C_2 may be arbitrary. The range of m , however, is restricted to $-1 < m \leq 1$ where the lower bound stems from the dynamical term and the existence of the integral in $\mathcal{F}(u)$. The upper bound also originates from the dynamical term ($m = 1$ requires in addition $a_0 > 0$).

Without any further specifications one may now study the asymptotic behaviour of the localized solitary waves $u(\xi)$ for $|\xi| \rightarrow \infty$. We investigate the behaviour of $\lim_{|\xi| \rightarrow \infty} \xi^q u(\xi)$ using (6) and the fact that $\xi \rightarrow \infty$ corresponds to $u \rightarrow 0$

$$\lim_{|\xi| \rightarrow \infty} \xi^q u(\xi) = \lim_{u \rightarrow 0} \left(\int_b^u \frac{dy}{\sqrt{-\mathcal{F}(y)}} \right)^q u. \quad (10)$$

Here we have chosen the lower bound of the integral to be b corresponding to $\xi = 0$.

In the limit $u \rightarrow 0$ we only have to consider the smallest powers of u in $\mathcal{F}(u)$ which, supposed the integration constant $C_1 \neq 0$, for $m < 0$ is u^{2-m} . The requirement that (10) asymptotically goes to zero then leads to the inequality

$$q < \frac{2}{|m|}.$$

Thus for any m in the range $-1 < m < 0$, $u(\xi)$ converges faster than ξ^{-2} . Precisely, for $-1 < m < 0$ one finds that asymptotically ($C_1 \neq 0$) $u(\xi) \sim \xi^{-2/|m|}$. For $0 < m < 1$ (10) leads to the condition

$$q < \frac{2}{1-m}.$$

Only for $m = 1$ one finds that $u(\xi)$ converges faster than any power of ξ . In this case the evaluation of (10) yields $\lim_{u \rightarrow 0} u \ln^q u = 0$ for any q . This property is related to the fact that $\mathcal{F}''(0) < 0$ for $m = 1$ and $\mathcal{F}''(0) = 0$ for $|m| < 1$.

We can easily deduce further properties of localized solitary wave solutions if in addition we restrict our considerations to $K(n, m)$ -type equations with $m > 0$. For $K(n, m)$ -type equations one has

$$\mathcal{V}(u) = nAu^{n-1} + v$$

that is

$$\mathcal{F}(u) = -\frac{2A}{m(m+n)}u^{n-m+2} - \frac{2v}{m(m+1)}u^{3-m}. \tag{11}$$

With (9) one finds

$$v > 0 \quad A < 0$$

$$b = \left(\frac{v(n+m)}{(m+1)|A|} \right)^{\frac{1}{n-1}}.$$

That is, the soliton moves in positive x -direction ($v > 0$) and the parameter A has to be smaller than zero. The amplitude of the soliton is proportional to the $(n - 1)$ th root of the velocity. For dark solitons, i.e. antisolitons, one instead finds $A > 0$ for n even and $A < 0$ otherwise.

Finally, one can deduce the basic properties of the width L of the soliton. The width is calculated at a certain height of the soliton, e.g., at half the maximum height:

$$L = 2 \int_{b/2}^b \frac{du}{\sqrt{-\mathcal{F}(u)}} = 2 \int_{b/2}^b \frac{du}{\sqrt{-\frac{2|A|}{m(n+m)}u^{n-m+2} + \frac{vu^{3-m}}{m(m+1)}}$$

$$= 2\sqrt{\frac{m(m+1)}{2vb^{1-m}}} \int_{1/2}^1 \frac{du}{u\sqrt{-u^{n-m} + u^{1-m}}}.$$

The last step follows with $(m + 1)|A|/(v(n + m)) = b^{1-n}$. Thus

$$L \sim \frac{1}{\sqrt{vb^{1-m}}} \sim \frac{1}{\sqrt{v^{\frac{n-m}{n-1}}}}$$

The soliton solutions of $K(n, 1)$ equations have the form (see section 3.2)

$$u(\xi) = (\pm)^n \left(\frac{(n+1)v}{2|A|} \right)^{\frac{1}{n-1}} \frac{1}{\cosh^{\frac{2}{n-1}} \left(\frac{(n-1)\sqrt{v}}{2} \xi \right)}.$$

2.2.2. Compact support solutions. In this section we will discuss localized solitary waves with compact support, i.e.

$$u(\xi) \begin{cases} \neq 0 & \text{if } \xi \in (\xi_1, \xi_2) \\ = 0 & \text{otherwise} \end{cases}$$

which shall be called *compact support solutions* (CSS). These solutions were introduced as *compactons* in [1]. $u(\xi)$ being a solution of (2) requires $\mathcal{D}(u(\xi))$ to be of class C_3 and $\mathcal{C}(u(\xi))$ and $u(\xi)$ to be at least of class C_1 at the boundary of the compact ξ interval. We will assume again that $\mathcal{D}(u) = u^m$, that $\mathcal{V}(u)$ is a continuous function of u and that the nonlinear dispersive evolution equation contains a dynamical term. With the considerations about the continuity, one finds in general that $\mathcal{F}(0) = u_\xi|_{\xi=\xi_1, \xi_2} = 0$ and that $C_1 = C_2 = 0$ and $m < 3$. Employing again the interpretation of the wavefunction $u(\xi)$ as a spacetime trajectory, one finds similar

to the soliton case in section 2.2.1 that the potential function $\mathcal{F}(u)$ must have a zero at $u = b$ (b denotes the amplitude of the CSS) in addition to the zero at $u = 0$ and is a continuous and integrable function in the interval $[0, b]$ with $\mathcal{F}(u) < 0$. Since unlike the soliton the CSS does not approach zero asymptotically but reaches it in a finite time ξ , the gradient of the potential function must not vanish at the boundaries of the u -interval, or, equivalently, $u_{\xi\xi} \neq 0$ for $\xi = \xi_1, \xi_2$. This leads to

$$\begin{aligned} \mathcal{F}(0) &= 0 & \mathcal{F}'(0) &< 0 \\ \mathcal{F}(b) &= 0 & \mathcal{F}'(b) &> 0. \end{aligned} \quad (12)$$

With the condition for the derivative of the potential function at $u = 0$ (this condition has been ignored in, e.g., [9]) one finally finds for CSS $m = 2$. Particularizing the considerations to $K(n, m)$ equations it further follows that

$$\begin{aligned} v &> 0 & A &< 0 \\ b &= \left(\frac{v(n+2)}{3|A|} \right)^{\frac{1}{n-1}}. \end{aligned} \quad (13)$$

Similar to the soliton case, the amplitude is proportional to the $(n-1)$ th root of the velocity. CSS with negative amplitude require both $v < 0$, i.e. they are moving in negative x -direction, and either $A > 0$ for n odd or $A < 0$ otherwise.

In the same way as for the solitons one finds for the widths of a CSS the following relation:

$$L \sim \sqrt{\frac{b}{v}} = \sqrt{\frac{1}{3}} \left(\frac{v}{3} \right)^{\frac{2-n}{2(n-1)}} \left(\frac{|A|}{(n+2)} \right)^{\frac{-1}{2(n-1)}}. \quad (14)$$

For $n = 2$ one thus finds the known result, that the width of the CSS is independent of its speed or its height [1].

As examples for CSS we give the solutions of the $K(2, 2)$ equation,

$$u(\xi) = \begin{cases} \frac{4v}{3|A|} \cos^2 \left(\frac{\sqrt{|A|}}{4} \xi \right) & \text{for } |\sqrt{|A|}\xi/4| \leq \pi/2 \\ 0 & \text{otherwise} \end{cases} \quad (15)$$

and of the $K(3, 2)$ equation,

$$u(\xi) = \begin{cases} \sqrt{\frac{5v}{3|A|}} \operatorname{sn}^2 \left(\left(\frac{v|A|}{240} \right)^{\frac{1}{4}} \xi \middle| -1 \right) & \text{for } 0 \leq \xi \leq \left(\frac{240}{v|A|} \right)^{\frac{1}{4}} \frac{2}{\sqrt{2}} K \left(\frac{1}{2} \right) \\ 0 & \text{otherwise} \end{cases}$$

both clearly showing properties (13) and (14) found from the general analysis of the potential function. In the latter solution, $\operatorname{sn}(x|m)$ denotes the Jacobian elliptic function $\operatorname{sn}(x|m) = \sin(\operatorname{am} u)$ with the Jacobi amplitude $\operatorname{am} u$ and $u = u(x, m) = \int_0^x d\theta (1 - m \sin^2 \theta)^{-1/2}$. $K(m)$ denotes the quarter period $u(\pi/2, m)$ [10].

Particularly for the $K(2, 2)$ -case we want to mention that a solution (15) added to a constant δ is again a solution of a $K(2, 2)$ equation. If $u(\xi)$ solves the $K(2, 2)$ equation $(u^2)_{\xi\xi\xi} = A(u^2)_{\xi} + vu_{\xi}$ with the potential function (see equation (11))

$$\mathcal{F}(u) = -\frac{A}{4}u^2 - \frac{v}{3}u$$

(with $C_1 = C_2 = 0$), then $U(\xi) = u(\xi) + \delta$ obeys, according to (8), the following potential representation:

$$(U_{\xi})^2 = \frac{A}{4}U^2 + \left(\frac{v}{3} + \frac{A}{2}\delta \right) U - \frac{A}{4}\delta^2 - \frac{v}{3}\delta \quad (16)$$

for the transformed equation $((U - \delta)^2)_{\xi\xi\xi} = A(U^2)_\xi + (v - 2A\delta)U_\xi$. Equation (16) may, however, as well be interpreted as the potential representation of a $K(2, 2)$ equation with $C_1 = -A\delta^2/4 - v\delta/3$. The solution $U(\xi)$ moves with the velocity $V = v + 3/2|A|\delta = 3/4|A|(b + 2\delta)$, v being the velocity of $u(\xi)$ [11, 12].

We want to make a last comment concerning the asymptotic behaviour of the solutions of the general equation (2), especially since some of the solutions of the $K(2, 2)$ equations are known to develop from smooth initial data to blow-up numerical solutions [1, 13]. In order to discuss the asymptotic behaviour of some general solutions of equation (2) we need to investigate the conditions under which this equation admits singularities or a break-down.

It is commonly known under which conditions strong singularities can occur in nonlinear systems spontaneously, or can develop from smooth initial data [14]. Moreover, it is rather the exception than the rule that compact supported initial data develop smoothly up to infinity and do not blow up after a finite time [1, 13].

One may use the characteristics procedure to prove or disprove the break-down of smooth solutions of equation (2) in the sense that some solutions become infinite in finite time (alternative stability conditions applied to the analysis of the stability of compact support solutions may be found in [15]). If we write equation (2) in the form

$$\Phi(t, x, u, p) = \mathcal{F}(u) + p^2 = 0$$

with the gradient $p = u_x$, the characteristics system associated with this equation is

$$\frac{dt}{ds} = 0 \quad \frac{dx}{ds} = \frac{\partial \Phi}{\partial p} = 2p \quad \frac{dp}{ds} = -p \frac{\partial \Phi}{\partial u} = -p\mathcal{F}'(u) \quad \frac{du}{ds} = p \frac{\partial \Phi}{\partial p} = 2p^2$$

where s is the parameter on the characteristics and $(t, x, u, 0, p)$ are prescribed in such a way that equation (2) holds at $s = 0$. There are two possible sources of singularities in this system. The Jacobian $\frac{\partial(t,x)}{\partial(s,u)}$, which in our case is proportional to $d\xi/ds$, may have vanishing determinant, so that one expects the solution to be multi-valued. In this case a shock wave formation can occur. Another possibility comes from the fact that u may be singular as a function of the x variable, a case in which we can have a blow-up singularity. For equation (2) the characteristics can be parametrized in the form

$$x = s \quad t = y \quad u = \frac{u_0(y)}{1 - su_0(y)}$$

where $y = u|_{t=0}$ is the initial data, that is the coordinate on the initial surface.

The results show that in the nonlinear dispersive, non-dissipative dynamical systems described above (dispersion function $\mathcal{D}(u) = u^m$ and $\mathcal{V}(u)$ being a continuous function of u containing a constant term in the power expansion corresponding to a dynamical term) soliton solutions may only occur if the degree of dispersion $-1 < m \leq 1$. For exactly linear dispersion ($m = 1$) one finds soliton solutions that asymptotically vanish faster than any power of ξ . The compact support solutions strictly require a quadratic dispersion ($m = 2$).

3. Classification of $K(n, m)$ equations by a point transformation

3.1. General considerations

In this section it will be shown that $K(n, m)$ -type equations with $m, n \geq 1$ can be transformed into other $K(N, M)$ -type equations with different arguments N and M . The applied (point) transformation uniquely connects the elements of certain sets of $K(n, m)$ equations. It constitutes an equivalence relation between those connected equations and thus serves as a tool to classify nonlinear dispersive evolution equations of $K(n, m)$ -type. The benefit of this

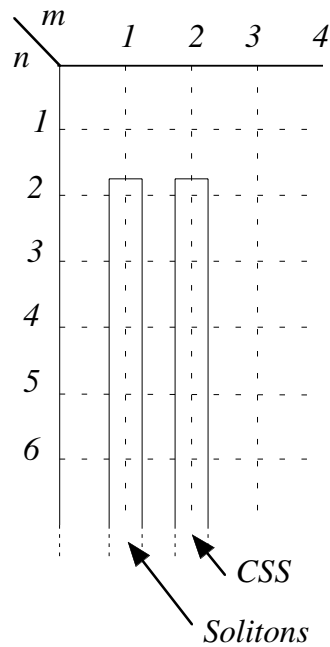


Figure 1. The chart of $K(n, m)$ equations. Any point on the lattice represents a $K(n, m)$ equation. Systems with linear dispersion $m = 1$ allow for soliton solutions whereas system with quadratic dispersion $m = 2$ may have compact support solutions.

classification is clearly the fact that the solutions to any element (equation) of an equivalence class can be traced back to the solution of the representative of this class. However, this transformation requires one to restrict the considerations to a certain subset of travelling solutions which is defined by fixing the initially arbitrary integration constants in the potential representation $C_1 = C_2 = 0$. In the case of $K(n, 1)$ or $K(n, 2)$ equations this is the necessary condition for solitons and compact support solutions which is summarized in figure 1 showing a chart of $K(n, m)$ equations.

The transformations of the potential representation, being a first-order ODE, are fully covered by the theory of point transformations [17]. Here we choose the transformation $u(\xi) = \omega^{\pm q}(\xi)$ with a real number q which transforms the potential representation of a nonlinear dispersive differential equation of type $K(n, m)$ into the potential representation of another $K(N, M)$ equation. The derivation of this transformation is carried out in the appendix. The transformation presented above allows for an entirely analytical treatment.

This simple point transformation suggests a diagram (figure 2) in which the $K(n, m)$ and the $K(N, M)$ equations are connected via the respective potential representations. Thus, if $u(\xi)$ is a solution of the $K(n, m)$ equation, $\omega(\xi)$ is a solution of the $K(N, M)$ equation. The subscript 0 of the potential function indicates the special choice $C_1 = C_2 = 0$.

M and N fulfil the relations

$$\left. \begin{aligned} M &= q(m-1) + 1 \\ N &= q(n-1) + 1 \end{aligned} \right\} \quad \text{for } u \rightarrow \omega^{+q} \quad (17)$$

$$\left. \begin{aligned} M &= q(n-m) + 1 \\ N &= q(n-1) + 1 \end{aligned} \right\} \quad \text{for } u \rightarrow \omega^{-q}. \quad (18)$$

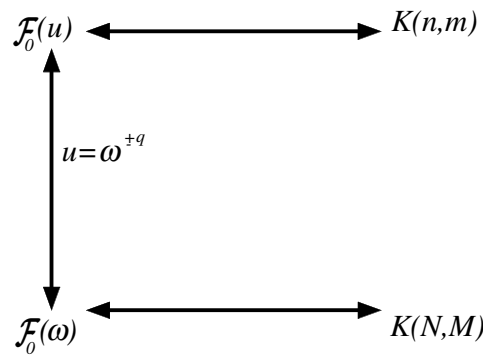


Figure 2. The point transformation of the potential representations of $K(n, m)$ -type equations.

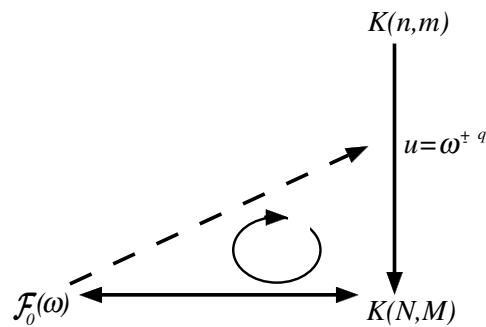


Figure 3. The transformation of $K(n, m)$ -type equations via a point transformation using the transformation property of the potential function as a consistency relation.

For M, N, m and n being integers, q can take on any rational number $q = (N - 1)/(n - 1)$, but in fact, the diagram remains valid for any value of q . Note, however, that although in this way the $K(n, m)$ equations may be transformed into equations with arbitrary, infinitesimal nonlinearities $n, m = 1 + \varepsilon$, the resulting equations do not represent analytic continuations of the linear cases and the respective solutions are not smoothly connected.

$K(n, m)$ equations with $m = (n + 1)/2$ are, as a peculiarity of the transformation, again transformed into equations with $M = (N + 1)/2$ by both (17) and (18).

The equation $K(n, m)$ can be transformed directly into $K(N, M)$. The diagram (figure 2) can thus be closed (see figure 3) by using the potential representation of the $K(N, M)$ equation as a consistency relation, indicated by the dashed line. To see this one calculates explicitly for case (17) with $u = \omega^{+q}$

$$(u^m)_{\xi\xi\xi} = A(u^n)_\xi + vu_\xi \quad (\omega^{mq})_{\xi\xi\xi} = (\omega^{q-1+M})_{\xi\xi\xi} = A(\omega^{nq})_\xi + v(\omega^q)_\xi. \quad (19)$$

Expanding $(\omega^{q-1+M})_{\xi\xi\xi}$, equation (19) can be rearranged as

$$(\omega^M)_{\xi\xi\xi} = -M(q - 1)(q + 2M - 4)\omega^{M-3}(\omega_\xi)^2\omega_\xi - \frac{3}{2}M(q - 1)\omega^{M-2}[(\omega_\xi)^2]_\xi + \frac{AM}{\omega^{q-1}mq}(\omega^{nq})_\xi + \frac{vM}{\omega^{q-1}mq}(\omega^q)_\xi \quad (20)$$

$$\stackrel{!}{=} A'(\omega^N)_\xi + v'\omega_\xi. \quad (21)$$

From the requirement that (20) equals a $K(N, M)$ equation with coefficients A' and v' (21), we infer also the existence of a potential representation

$$(\omega_\xi)^2 = -\mathcal{F}(\omega) = \frac{2A'}{M(M+N)}\omega^{N-M+2} + \frac{2v'}{M(+1)}\omega^{3-M} \quad [(\omega_\xi)^2]_\xi = -\mathcal{F}'(\omega)\omega_\xi.$$

Here we have again made the special choice $C_1 = C_2 = 0$ which is a necessary condition for the general case (see also the appendix). Inserting this condition for the squared derivative of the transformed wavefunction ω into (20), one gets

$$\begin{aligned} (\omega^M)_{\xi\xi\xi} &= \frac{MA n}{m}\omega^{N-1}\omega_\xi + \frac{Mv}{m}\omega_\xi - (q-1)(q+2M-4) \left(\frac{2A'}{(M+N)}\omega^{N-1} + \frac{2v'}{(M+1)} \right) \omega_\xi \\ &\quad - 3(q-1) \left(\frac{A'(N-M+2)}{(M+N)}\omega^{N-1} + \frac{v'(3-M)}{(M+1)} \right) \omega_\xi. \end{aligned}$$

Comparing the last equation with (21) yields the conditions for the new coefficients:

$$\begin{aligned} \frac{Mn}{m}A - \frac{2(q-1)(q+2M-4)}{(M+N)}A' - \frac{3(q-1)(N-M+2)}{(M+N)}A' &= NA' \\ \frac{M}{m}v - \frac{2(q-1)(q+2M-4)}{(M+1)}v' - \frac{3(q-1)(3-M)}{(M+1)}v' &= v' \end{aligned}$$

leading to

$$A' = \frac{AM(M+N)}{q^2m(m+n)} \quad (22)$$

$$v' = \frac{vM(M+1)}{q^2m(m+1)}. \quad (23)$$

Similarly, one obtains for case (18) a $K(N, M)$ equation with coefficients

$$A' = \frac{vM(M+N)}{q^2m(m+1)} \quad (24)$$

$$v' = \frac{AM(M+1)}{q^2m(m+n)}. \quad (25)$$

One finds here that transformation (18) interchanges the nonlinear convection term and the dynamical term. Transformation (18) is thus a purely mathematical relation between the ODEs considered with any physical implication being removed. The fact that the transformation does not seem to preserve invariants and integrability properties may be based on this circumstance. The KdV equation, having an infinite countable set of invariants, can for example be transformed into the $K(2, 2)$ equation (see the next section), having only three invariants [1, 16].

To illustrate this result, a chart of the $K(n, m)$ equations shown in figure 4 gives three sets of $K(n, m)$ equations connected by the point transformation discussed, i.e. three different equivalence classes. In fact, under the assumption of certain kinds of travelling solutions, the latter being specified by a particular choice of integration constants in the potential representation, any $K(n, m)$ equation belongs to a certain equivalence class whose representative is characterized by m and n with $(m-1)$ and $(n-1)$ having no common divisors and $n \geq 2m-1$. The latter restriction is based on the fact that any $K(n, m)$ equation with $n < 2m-1$ is connected to a $K(n, m)$ equation with $n > 2m-1$ through transformation (18).

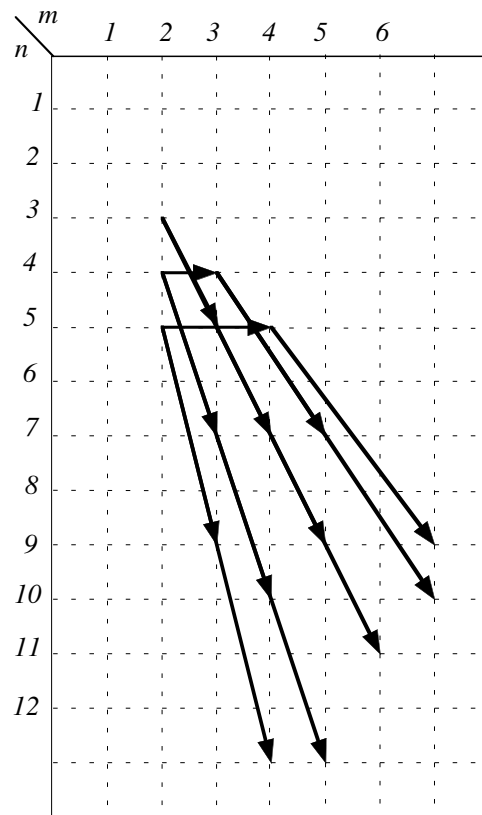


Figure 4. The chart of $K(n, m)$ equations showing several connected sets (equivalence classes).

Finally, we want to mention a peculiarity arising for $m = 1, q = 2$. In this case the first term on the rhs of equation (20) does not contribute and only the derivative of the potential function $\mathcal{F}(\omega)$ enters the transformation of the $K(n, 1)$ equation. This allows the potential function $\mathcal{F}(\omega)$ to contain an arbitrary constant, corresponding to the term $C_2\omega^{2-2m} = C_2$ in (7), and according to the transformation $u = \omega^2, \mathcal{F}(u)$ may additionally contain the term $4C_2u^{2-m} = 4C_2u$. Here $\mathcal{F}(u)$ belongs to a $K(n, 1)$ equation with the parameters A and v and, accordingly, $\mathcal{F}(\omega)$ is the potential function of a $K(2n - 1, 1)$ equation with the parameters $A' = n/(2(n + 1))A$ and $v' = v/4$. In this way, pairs of $K(n, 1)$ equations become particularly connected. In [6] this property has been addressed especially for the pair $K(3, 1) - K(5, 1)$.

3.2. Example: the equivalence class of the KdV equation

The KdV equation reads

$$u_{\xi\xi\xi} = A(u^2)_{\xi} + vu_{\xi}$$

According to section 2.2, we choose $A < 0$ to have soliton solutions. A soliton solution has the form

$$u(\xi) = \frac{3v}{2|A|} \frac{1}{\cosh^2\left(\frac{\sqrt{v}}{2}\xi\right)}$$

Using the results from the previous section one can immediately state the solutions to any $K(N, 1)$ or $K(N, N)$ equation (see figure 5). On the one hand, one finds that with the transformation $u = w^{+q}$ solutions to $K(N, 1) = K(q + 1, 1)$ equations read (the parameters v and A have been expressed by the transformed parameters v' and A')

$$w(\xi) = (\pm)^N \left(\frac{(N+1)v'}{2|A'|} \right)^{\frac{1}{N-1}} \frac{1}{\cosh^{\frac{2}{N-1}} \left(\frac{(N-1)\sqrt{v'}\xi}{2} \right)}.$$

As indicated by the resulting factor $(\pm)^N$ on the rhs one finds immediately that $K(N, 1)$ equations with a symmetric potential function, i.e. with odd N have both soliton and antisoliton solutions. On the other hand, the solutions for the resulting $K(N, N) = K(q + 1, q + 1)$ equations of the transformation $u = w^{-q}$ read

$$w(\xi) = (\pm)^N \left(\frac{2|v'|N}{(N+1)A'} \right)^{\frac{1}{N-1}} \cosh^{\frac{2}{N-1}} \left(\frac{\sqrt{A'}(N-1)}{2N} \xi \right). \quad (26)$$

To compare this result with the literature, e.g. [18], one needs to recall that the potential function belonging to the transformed $K(N, N)$ equation is just the negative of the potential function of $K(n, n)$ equations usually used in the literature. Changing the signs accordingly results in the change $\xi \rightarrow i\xi$, i.e. the hyperbolic cosine in equation (26) is changed to a trigonometric cosine as in [18]. For $m = n = 2$ the solution (26) can be compactified. With the appropriate changes it represents the CSS (compare section 2.2)

$$w(\xi) = \begin{cases} \frac{4v'}{3|A'|} \cos^2 \left(\frac{\sqrt{|A'|}}{4} \xi \right) & \text{for } |\sqrt{|A'|}\xi/4| \leq \pi \\ 0 & \text{otherwise.} \end{cases} \quad (27)$$

Finally, a certain nonlinear dispersive evolution equation of non- $K(n, m)$ -type will be investigated that, nevertheless, can be considered as an element of the KdV equivalence class. The equation under consideration reads

$$\beta(u^2)_{\xi\xi\xi} + \varepsilon u_{\xi\xi\xi} = -\alpha(u^2)_{\xi} - u_t. \quad (28)$$

With $\beta(u^2)_{\xi\xi\xi} + \varepsilon u_{\xi\xi\xi} = \beta((u + \varepsilon/(2\beta))^2)_{\xi\xi\xi}$ and $u_t = -vu_{\xi}$ the corresponding potential representation is found with (8). Basically we have dealt with this problem already in section 2.2. We may thus immediately assume the solution to have the form of a CSS (27) added to a constant. In particular, the potential representation reads

$$(u_{\xi})^2 = -\frac{\alpha}{4\beta} u^2 + \left(-\frac{\alpha\varepsilon}{4\beta^2} + \frac{(\alpha\varepsilon + v\beta)}{3\beta^2} \right) u - C_1 - \frac{\alpha\varepsilon^2}{16\beta^3} + \frac{\varepsilon(\alpha\varepsilon + v\beta)}{6\beta^3}$$

with C_2 set equal to zero to avoid singularities of the potential function. Setting

$$U(\xi) = u(\xi) + \frac{\varepsilon}{2\beta} - \frac{2(\alpha\varepsilon + v\beta)}{3\alpha\beta} + \sqrt{\frac{4(\alpha\varepsilon + v\beta)^2}{9\alpha^2\beta^2} - C_1}$$

the potential representation can be transformed into the form

$$(U_{\xi})^2 = -\frac{\alpha}{4\beta} U^2 + \sqrt{\frac{(\alpha\varepsilon + v\beta)^2}{9\beta^4} - \frac{\alpha^2 C_1}{4\beta^2}} U.$$

With the preceding analysis the solution for $U(\xi)$ and thus for $u(\xi)$ can be given immediately

$$u(\xi) = \frac{4}{3\alpha\beta} \sqrt{(\alpha\varepsilon + v\beta)^2 - 9/4\alpha^2\beta^2 C_1} \cos^2 \left(\frac{\sqrt{\alpha/\beta}}{4} \xi \right) - \frac{\varepsilon}{2\beta} + \frac{2(\alpha\varepsilon + v\beta)}{3\alpha\beta} - \sqrt{\frac{4(\alpha\varepsilon + v\beta)^2}{9\alpha^2\beta^2} - C_1}.$$

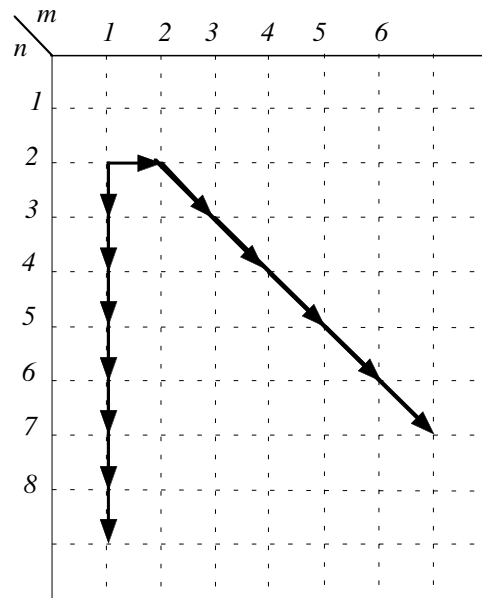


Figure 5. The equivalence class of connected $K(n, m)$ equations with $K(2, 1)$ as its representative, i.e. the KdV equation.

The solution $u(\xi)$ thus has the amplitude

$$b = \frac{4}{3\alpha\beta} \sqrt{(\alpha\varepsilon + v\beta)^2 - 9/4\alpha^2\beta^2 C_1}$$

and width

$$L = \frac{4}{\sqrt{\alpha/\beta}}.$$

It moves with the speed

$$v' = 3 \left(-\frac{\alpha\varepsilon}{4\beta^2} + \frac{(\alpha\varepsilon + v\beta)}{3\beta^2} \right)$$

and is shifted by

$$\delta = -\frac{\varepsilon}{2\beta} + \frac{2(\alpha\varepsilon + v\beta)}{3\alpha\beta} - \sqrt{\frac{4(\alpha\varepsilon + v\beta)^2}{9\alpha^2\beta^2} - C_1}.$$

For $C_1 = 0$, these equations simplify enormously and one finds the simple relation for the speed

$$v' = \frac{3\alpha}{4\beta} \left(\frac{\varepsilon}{\beta} - b \right).$$

Here one finds that $b_{\text{crit}} = \varepsilon/\beta$ constitutes a critical amplitude. Solutions with $b > b_{\text{crit}}$ move to the left whereas solutions with $b < b_{\text{crit}}$ move to the right. Solutions with $b = b_{\text{crit}}$ are at rest. This property of travelling modes in systems having both quadratic and linear dispersions is documented in [11, 12, 19].

4. Summary

In this paper we have presented a potential picture for nonlinear dispersive wave equations. The potential picture provides a simplified representation of the original wave equation in terms of a nonlinear first-order ordinary differential equation which is valid for the set of travelling solutions. The potential representation allows for an easy and intuitive way to identify different types of possible solutions. It proves to be extremely useful for the examination of nonlinear dynamical systems, since it provides direct access to various properties of the solutions without a need to solve the underlying nonlinear dispersive wave equation. Choosing the dispersion function to be a power of the wavefunction one finds from a general investigation of the potential picture that for localized solitary waves the degree of dispersion is restricted to $-1 < m \leq 1$ whereas compact support solutions may only arise as possible modes in systems with quadratic dispersion. The potential representation allows the determination of the asymptotic behaviour of solitary waves; furthermore, the specification of this concept to the so-called $K(n, m)$ equations—a certain kind of nonlinear dispersive wave equations—directly gives the relations between the amplitude of the solitary wave and its speed or the width of the wave and its speed and height, respectively. Particularly for the compact support solution of the $K(2, 2)$ equation the width of the wave is independent of its speed or its height.

Furthermore, it has been shown that the potential representations of a certain $K(n, m)$ equation can be transformed into the respective potential representation of another $K(n, m)$ -type equation by a simple point transformation. Using the potential representation as a consistency relation for the derivative of the wavefunction, $K(n, m)$ equations can be transformed directly into another. In this way the $K(n, m)$ equations are divided into equivalence classes, each containing the set of equations that are connected via the considered point transformation. This transformation requires a further restriction of the admissible solutions. In addition to the requirement of focusing on travelling solutions, the solutions are specified by fixing the initially arbitrary integration constants to zero. An important property of point transformations is their invertibility. All elements of an equivalent class are uniquely connected with each other.

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Appendix. Derivation of the point transformation

We start from the general potential representation of $K(n, m)$ equations

$$(u_\xi)^2 = \frac{2A}{m(m+n)} u^{n-m+2} + \frac{2v}{m(m+1)} u^{3-m} + C_1 u^{2-m} + C_2 u^{2-2m}.$$

Assuming the point transformation $u = \theta(\omega)$ one is lead to

$$(\omega_\xi)^2 = \frac{2A}{m(m+n)} \frac{\theta^{n-m+2}}{\theta^2} + \frac{2v}{m(m+1)} \frac{\theta^{3-m}}{\theta^2} + C_1 \frac{\theta^{2-m}}{\theta^2} + C_2 \frac{\theta^{2-2m}}{\theta^2}. \quad (29)$$

Equation (29) is again the potential function of a $K(N, M)$ equation (with different arguments N and M), i.e.

$$(\omega_\xi)^2 = \frac{2A'}{M(M+N)}\omega^{N-M+2} + \frac{2v'}{M(M+1)}\omega^{3-M} + C'_1\omega^{2-M} + C'_2\omega^{2-2M}.$$

The four terms on the rhs constitute (in combination with (29)) four differential equations that determine the transformation $\theta(\omega)$. With only two free parameters M and N they can, in general, not be fulfilled simultaneously so that we have to assume $C_1 = C_2 = 0$. One is left with

$$\frac{\theta^{n-m+2}}{\theta'^2} = \mathcal{A}\omega^{N-M+2} \quad \frac{\theta^{3-m}}{\theta'^2} = \mathcal{B}\omega^{3-M} \tag{30}$$

with

$$A' = \frac{\mathcal{A}M(M+N)}{m(m+n)}A \quad v' = \frac{\mathcal{B}M(M+1)}{m(m+1)}v$$

or alternatively

$$\frac{\theta^{n-m+2}}{\theta'^2} = \mathcal{B}\omega^{3-M} \quad \frac{\theta^{3-m}}{\theta'^2} = \mathcal{A}\omega^{N-M+2} \tag{31}$$

with

$$v' = \frac{\mathcal{B}M(M+1)}{m(m+n)}A \quad A' = \frac{\mathcal{A}M(M+N)}{m(m+1)}v.$$

In the following, we will only deal with transformations (30). One finds in general

$$\theta = \left(\frac{\mathcal{A}}{\mathcal{B}}\omega^{N-1}\right)^{\frac{1}{n-1}}. \tag{32}$$

Further we first consider the cases covered by $m \neq 1, n \neq m, M \neq 1, N \neq M$. Straightforward integration of equations (30) leads to

$$\begin{aligned} \theta &= \left(\frac{1}{\mathcal{A}}\left(\frac{m-n}{M-N}\right)^2\omega^{M-N}\right)^{\frac{1}{m-n}} \\ &= \left(\frac{1}{\mathcal{B}}\left(\frac{m-1}{M-1}\right)^2\omega^{M-1}\right)^{\frac{1}{m-1}}. \end{aligned}$$

These equations together with (32) yield the conditions

$$\frac{N-1}{n-1} = \frac{M-N}{m-n} = \frac{M-1}{m-1} \quad \frac{1}{\mathcal{A}}\left(\frac{m-n}{M-N}\right)^2 = \frac{1}{\mathcal{B}}\left(\frac{m-1}{M-1}\right)^2 = 1 \quad \mathcal{A} = \mathcal{B}$$

that are fulfilled by

$$M-N = q(m-n) \quad M-1 = q(m-1) \quad \mathcal{A} = \mathcal{B} = \frac{1}{q^2}.$$

Therefore, the parameters M and N of the $K(N, M)$ equation belonging to the transformed potential representation are

$$M = q(m-1) + 1 \quad N = q(n-1) + 1. \tag{33}$$

Transformation (30) finally reads

$$\theta = \omega^{+q}. \tag{34}$$

The sign of the exponent is fixed by the requirement $M, N > 0$.

Let us now turn to the cases in which either $m = 1$ or $m = n$ or $M = 1$ or $M = N$. One finds that the combinations $m = M = 1$ or $m = n = M = N = 0$ in case (33) are likewise covered by (34). The remaining four combinations, i.e. $m = 1$ and $N \neq M$, $m = 1$ and $N = M$, $m = n$ and $N \neq M$, or $m = n$ and $M = 1$, respectively, do not lead to solutions of (30) that are in agreement with (32) and can thus be discarded. Equivalently, one finds for the alternatively possible transformation (31)

$$M = q(n - m) + 1 \quad N = q(n - 1) + 1.$$

Here the transformation reads

$$\theta = \omega^{-q}.$$

In addition to the requirement $M, N > 0$ one has to assume in this case that $n \geq m$.

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